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COMPARISON BETWEEN THE CONVERGENCE OF POWER AND MITTAG-LEFFLER SERIES

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1 Introduction

In this paper we consider series in functions of Mittag-Leffler type, namely in $z^{\alpha}E_{\alpha}(z)$ and $z^{\beta}E_{\alpha,\,\beta}(z)$. We study their convergence, more precisely, we determine where these series converge and where do not, where the convergence is uniform and where is not. The radius of their disks of convergence is found and the behaviour on the boundaries of these domains is studied. Namely, we provide theorems of Cauchy-Hadamard, Abel and Fatou type and compare them with the classical results for the more popular power series. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler type functions and their generalizations (as for example in Kiryakova [2] and Sandev, Tomovski and Dubbeldam [10]).

For our purpose we need the definitions of the Mittag-Leffler (M-L) functions

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\ \beta>0, \tag{1.1}$$

and introduce the following auxiliary Mittag-Leffler functions, related to (1.1):

$$\widetilde{E}_0(z)=1;\quad \widetilde{E}_n(z)=z^nE_n(z),\ n\in\mathbb{N},$$

$$\widetilde{E}_{\alpha,0}(z) = 1; \quad \widetilde{E}_{\alpha,n}(z) = \Gamma(n) z^n E_{\alpha,n}(z),\ n \in \mathbb{N}; \quad \alpha > 0, \tag{1.2}$$

where $\widetilde{E}_0(z)$, $\widetilde{E}_{0,\beta}(z)$) and $\widetilde{E}_{\alpha,0}(z)$ are added just for completeness.
Then we consider simultaneously the series in these functions (1.2), respectively:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_n(z), \qquad \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z), \tag{1.3}$$

and the power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{1.4}$$

with same complex coefficients a_n $(n = 0, 1, 2, ...)$.

We give some results for series of the kind (1.3) in the general case, proven in [7], and consider their behaviour on an arc of the unit circle $|z| = 1$, all the points of which (including the ends) are regular to the sum of the series. Such type of results have been also obtained for series in other special functions, for example, for series in Laguerre and Hermite polynomials (in Rusev [9]), and in previous author's papers, resp. for systems of Bessel functions ([4]) and of some other special functions of fractional calculus which are fractional indices analogues of the Bessel functions, as the Bessel-Maitland functions ([5], [6]) and the multi-index Mittag-Leffler functions (in the sense of Kiryakova [1]), see e.g. [8].

2 Theorems of Cauchy-Hadamard and Abel type

In the beginning we give a theorem of Cauchy-Hadamard type and a corollary for every one of the above mentioned series.

Theorem 2.1. (of Cauchy-Hadamard type). *The domain of convergence of each of the series (1.3) and (1.4) with complex coefficients a_n is the disk $|z| < R$ with a radius of convergence $R = 1/\Lambda$, where*

$$\Lambda = \limsup_{n\to\infty} (\ |a_n|\ )^{1/n}. \tag{2.1}$$

More precisely, the series (1.3) and (1.4) are absolutely convergent on the disk $|z| < R$ and divergent on the domain $|z| > R$. The cases $\Lambda = 0$ and $\Lambda = \infty$ can be included in the general case, provided $1/\Lambda$ means ∞, respectively 0.

Corollary 2.1.1. *Let each of the series (1.3) and (1.4) converge at the point $z_0 \neq 0$. Then it is absolutely convergent on the disk $D = \{z : |z| < |z_0|,\ z \in \mathbb{C}\}$. Inside of the disk $|z| < 1/\Lambda = R$, i.e. on each closed disk $|z| \le r < R$ (Λ defined by (2.1)), the convergence is uniform.*

So, it turns out that each of the M-L series (1.3), as well as the power series (1.4), converges in a disk with one and the same radius R, and diverges on its outside. Moreover, on each closed disk $|z| \le r < R$, the convergence is uniform. The very disk of convergence is not obligatory a domain of uniform convergence and on its boundary the series may even be divergent.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$, g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0, and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle.

The next theorem refers to the uniform convergence on the set d_φ and the convergence at the point z_0, provided $|z| < R$ and $z \in g_\varphi$.

Theorem 2.2. (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be the real number defined by (2.1), $0 < \Lambda < \infty$. Let $K = \{z : z \in \mathbb{C}, |z| < R, R = 1/\Lambda\}$. If $f(z)$, $g(z)$, $h(z;\alpha)$ are the sums respectively of the series (1.4) and (1.3) on the domain K, i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z), \quad h(z;\alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\, n}(z); \quad z \in K,$$

and these series converge at the point z_0 of the boundary of K, then the series (1.3) and (1.4) are uniformly convergent on the domain d_φ. and

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n z_0^n, \quad \lim_{z\to z_0} g(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z_0),$$

$$\lim_{z\to z_0} h(z;\ \alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\, n}(z_0),$$

provided $|z| < R$ and $z \in g_\varphi$.

The proofs of Theorems 2.1 and 2.2, except for the uniformity, are given in [7], while *the proofs* of Corollary 2.1.1 and the uniformity will be given elsewhere.

3 Fatou type theorems

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\limsup\limits_{n\to\infty} (\ |a_n|\ )^{1/n} = 1/R$, and $f(z)$ be the sum of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ on the open disk $U(0;R) = \{z : z \in \mathbb{C},\ |z| < R\}$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U(0;R).$$

Definition 3.1. *A point $z_0 \in \partial U(0;R)$ is called regular for the function f if there exist a neighbourhood $U(z_0;\rho)$ and a function $f_{z_0}^{*} \in \mathcal{H}(U(z_0;\rho))$ (the space of complex-valued functions, holomorphic in the set $U(z_0;\rho)$), such that $f_{z_0}^{*}(z) = f(z)$ for $z \in U(z_0;\rho) \cap U(0;R)$.*

By this definition it follows that the set of regular points of the power series is an open subset of the circle $C(0;R) = \partial U(0;R)$ with respect to the relative topology on $\partial U(0;R)$, i.e. the topology induced by that of $\mathbb{C}$.

In general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points. For example, the power series $\sum\limits_{n=0}^{\infty} z^n$ is divergent at each point of the circle $C(0;1)$ regardless of the fact that all the points of this circle, except $z = 1$, are regular for its sum. The series $\sum\limits_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z = 1$, is a singular (i.e. not regular) for its sum. But under additional conditions, imposed on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation do exists (see for details [3], vol.1, ch. 3, §7, 7.3, p.357).

Proposition referring to the above discussed properties holds also for series in the Laguerre and Hermite systems (see e.g. [9]). Here we give such type of theorem for series in Mittag-Leffler systems.

First, we remind the classical results in this direction.

Theorem 3.1. (of Fatou). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition $\limsup\limits_{n\to\infty} (\,|a_n|\,)^{1/n} = 1$ and $f(z)$ be the sum of the power series (1.4) on the disk $D = \{z : z \in \mathbb{C},\ |z| < 1\}$, i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in D.$$

Let σ be an arbitrary arc of the unit circle $|z| = 1$ with all its points (including the ends) regular to the function f (resp. g or h). Let $\lim\limits_{n\to\infty} a_n = 0$. Then the power series (1.4) converges, even uniformly, on the arc σ.

A statement analogous to this classical result, concerning the considered series of the type (1.3), is given below.

Theorem 3.2. (of Fatou type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition $\limsup\limits_{n\to\infty} (\,|a_n|\,)^{1/n} = 1$ and $g(z)$, $h(z;\alpha)$ be the sums respectively of the first and second of the series (1.3) on the disk $D = \{z : z \in \mathbb{C},\ |z| < 1\}$, i.e.*

$$g(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z),\ h(z;\ \alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\,n}(z);\ z \in D.$$

Let σ be an arbitrary arc of the unit circle $|z| = 1$ with all its points (including the ends) regular to the function g (or resp. h). Let $\lim\limits_{n\to\infty} a_n = 0$ and $\widetilde{E}_n(z) \neq 0$ (respectively $\widetilde{E}_{\alpha,\,n}(z) \neq 0$) for $z \in \sigma$. Then the first (resp. second) of the series (1.3) converges, even uniformly, on the arc σ.

The proof will be given elsewhere.

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